**Applications of Derivatives**

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## Extreme Values of Functions

Let be a function with domain . has an absolute maximum value on at a point if for all values of in , and an absolute minimum value on at a point if for all values of in . The maximum and minimum values are called the extreme values of the function.

### The Extreme Value Theorem

Statement: If is continuous on a closed interval , then attains both an absolute maximum value and an absolute minimum value in , i.e. there are numbers and in with and for which for all in .

### Local Extreme Values

A function has a local maximum value at a point within its domain if for all values of in lying in some open interaval containing . Similarly, a function has a local maximum value at a point within its domain if for all values of in lying in some open interval containing .

### The First Derivative Theorem for Local Extreme Values

Statement: If has a local maximum or minimum value at an interior point of its domain and if is defined at , then .

Example 2

Find the absolute maximum or minimum value of on .

is defined for all values of in .

For critical point, .

The given function has an absolute maximum value at and an absolute minimum value at .

Example 3

Find the absolute maximum and minimum values of on .

At the critical point, .

is not in .

The given function has an absolute maximum value of at and an absolute minimum value of at .

Exercise 4.1

21 – 36: Find the absolute maximum and minimum values of .

21.

This function does not have a critical point.

The given function has an absolute minimum value of at and an absolute maximum value of at .

23.

At the critical point,

The given function has an absolute minimum value of at and an absolute maximum value of at .

49 – 58: Find the extreme values (absolute and local) of the functions and where they occur.

50.

For critical point,

To check the local maximum and minimum, we consider two values of before and after the critical points.

For , let and .

For ,

For ,

For ,

Since both and are greater than , the function has a local minimum value of at .

For , let and .

For , .

For , .

For , .

Since both and are less than , the function has a local maximum value of at .

## The Mean Value Theorem

### Rolle’s Theorem

If is a continuous function at every point of its closed interval and differentiable at every point of its open interval , and if , then there exists at least one value of , say , between and such that .

Verify Rolle’s Theorem for the function in the interval .

Here, is continuous in the interval , differentiable in the interval and . Hence, according to Rolle’s Theorem, there exists at least one point between and , where .

Since lies in the interval , the theorem is verified.

### The Mean Value Theorem

If is a continuous function at every point in the closed interval and is differentiable at every point of the open interval , then there exists a point, say , between and , where .

Geometrically, the Mean Value Theorem says that there is a point between and where the tangent line is parallel to the chord .

Exercise 5

Verify the Mean Value Theorem for in the interval .

Here, is continuous in the interval and differentiable in the interval . The Mean Value Theorem states that hence, there is a value, , for which

and

Since both values of lie between and , the theorem is valid.

Exercise 6

Verify the Mean Value Theorem for the equation .

(N.A.)

Since lies between and lies between , the Mean Value Theorem is verified.

## Monotonic Functions

A function that is increasing or decreasing on an interval is said to be a monotonic function.

Increasing or Decreasing Functions: Suppose is continuous in the closed interval and differentiable in the open interval . Then is said to be increasing on if at each point and is said to be decreasing on if at each point . This is called the 1st Derivative Test for Monotonic Functions.

Example 1

Find the critical points of and identify the intervals on which is increasing or decreasing.

At critical points, .

and

The critical points and subdivide the domain on which is either positive or negative to create non-overlapping open intervals , and .

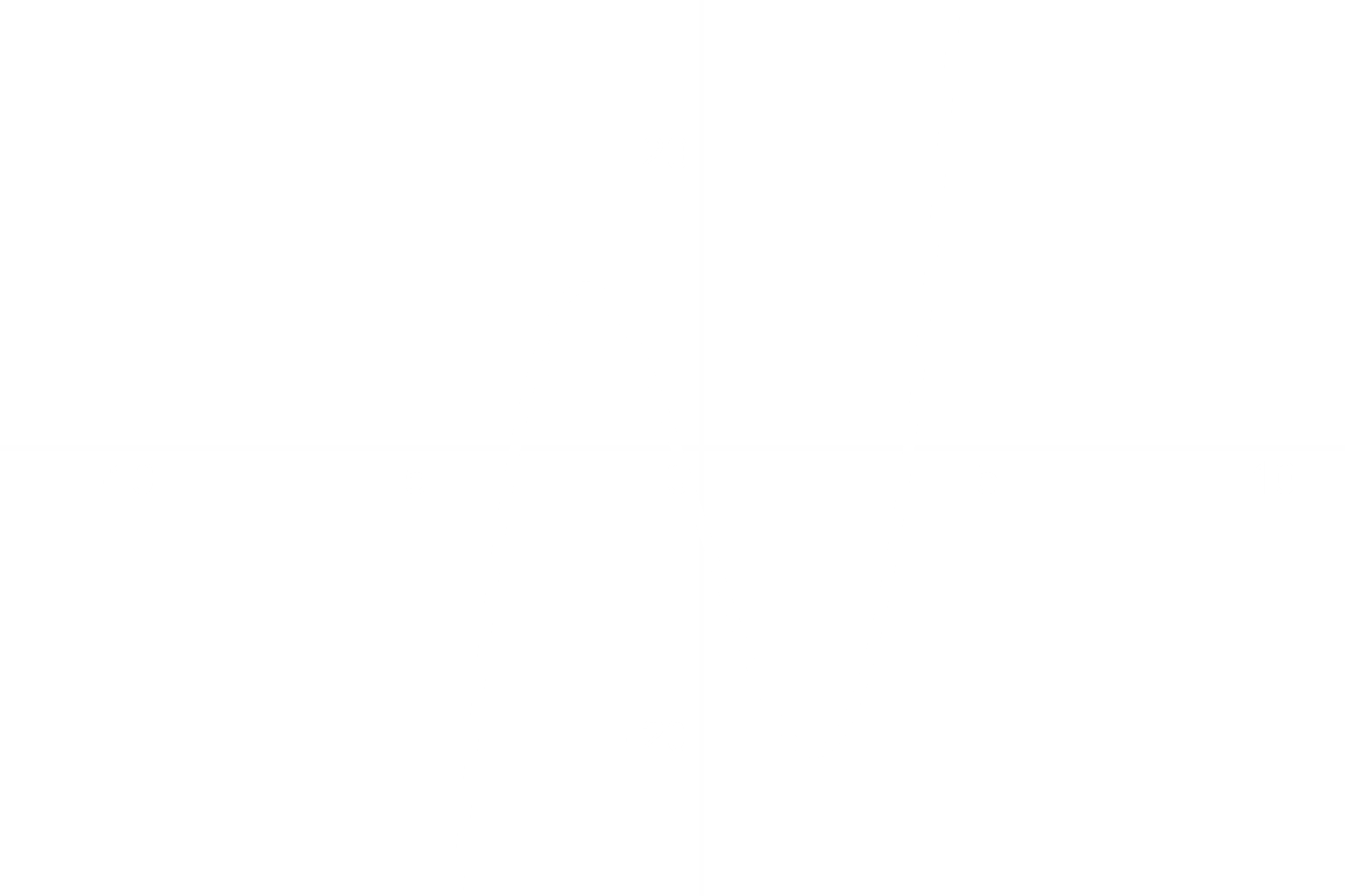
On the interval , let .

On the interval , let .

On the interval , let .

On the interval , is increasing.

On the interval , is decreasing.

On the interval , is increasing.

### 1st Derivative Test for Local Extremes

If changes from negative to positive at a point , then has a local minimum at . If changes from positive to negative at a point , then has a local maximum at . If does not change at a point , then has no local extreme at .

In the previous example, since changes from positive to negative at , the function has a local maximum value at .

The local maximum value is .

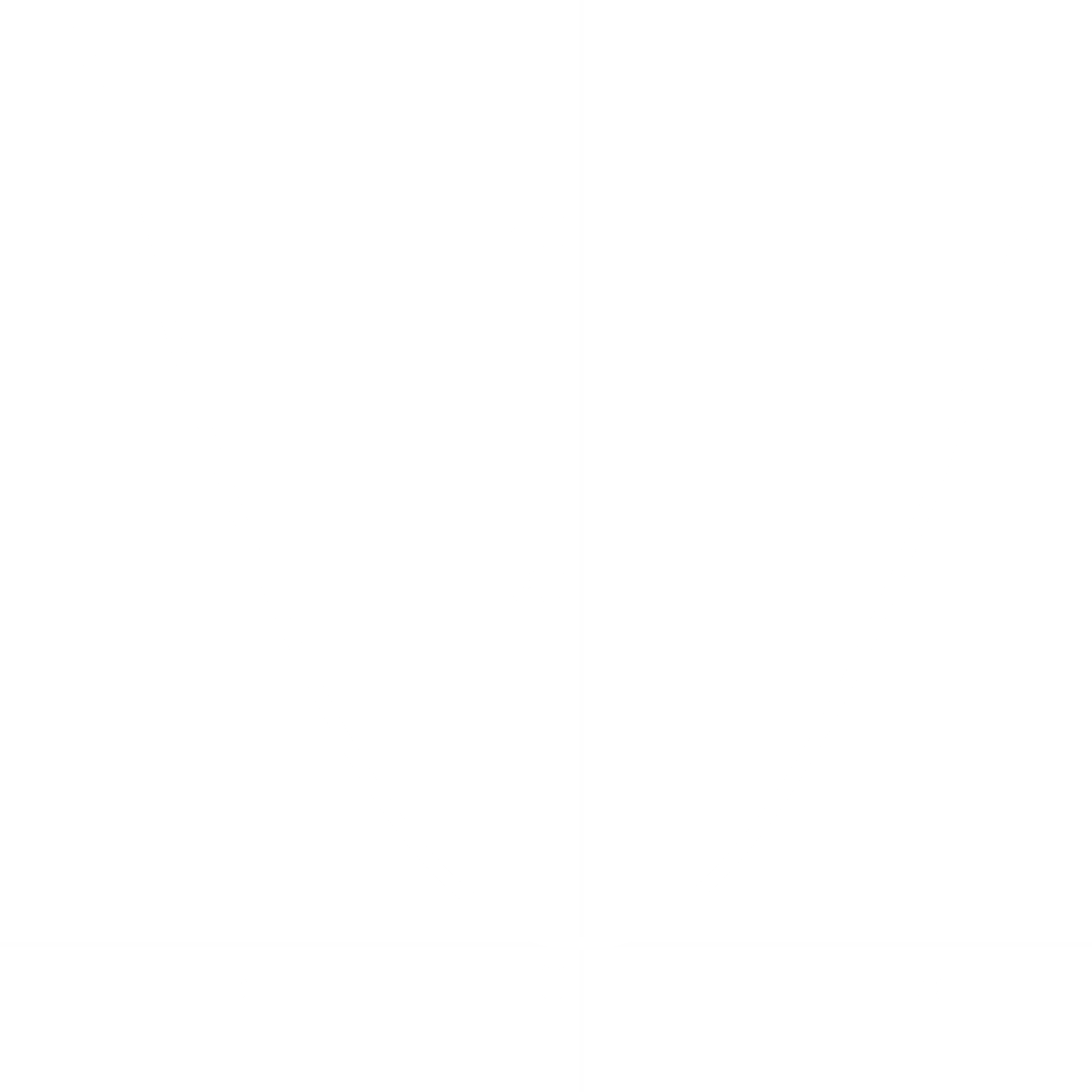
changes from negative to positive at , so has a local minimum value at .

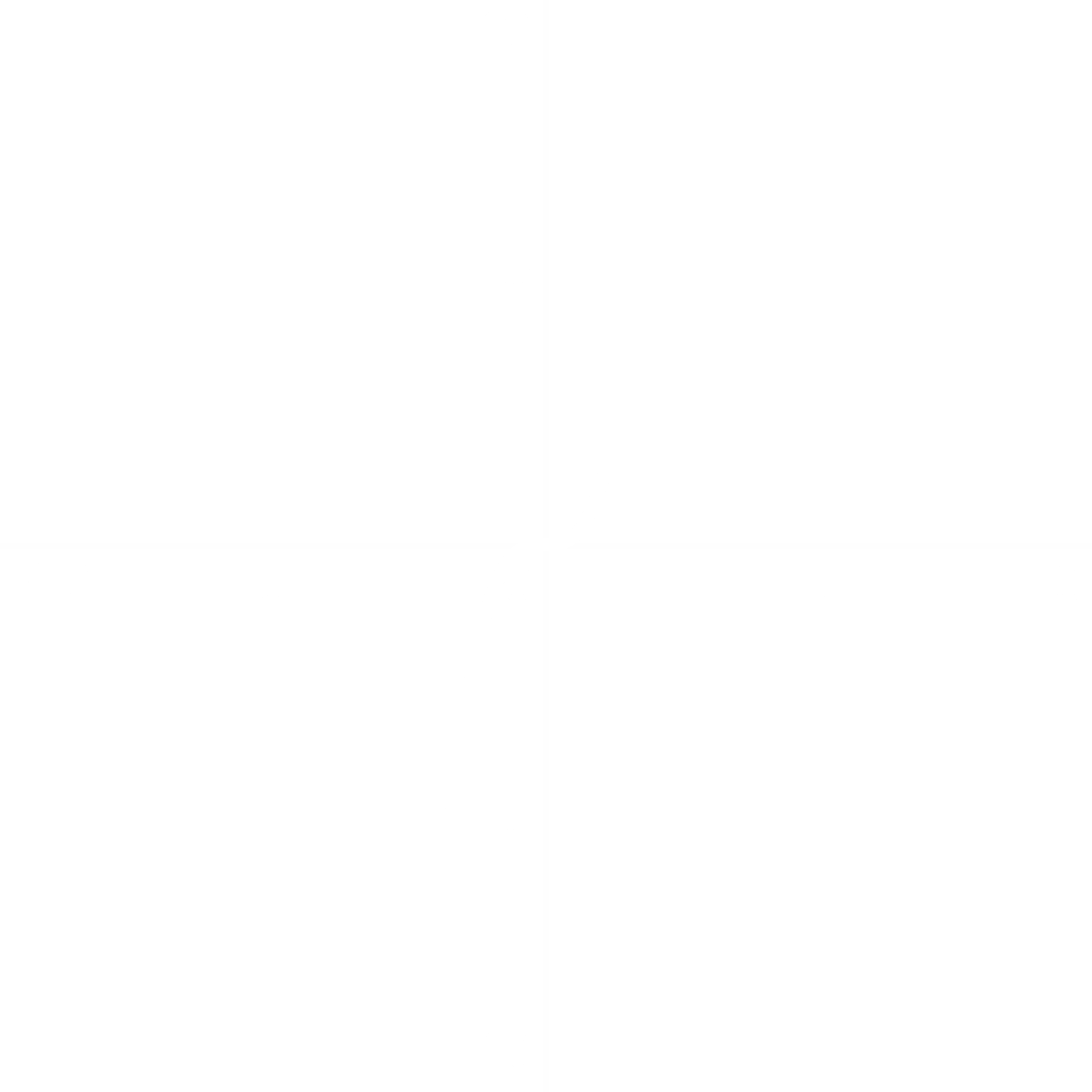
The local minimum value is .

## Concavity and Curve Sketching

The graph of a differentiable function is concave up on an open interval if is increasing (i.e. ), and concave down on an open interval if is decreasing (i.e. ).

Example:

The curve is concave up on as .

The curve is concave down on as and concave up on

as .

### Point of Inflection

Point where the graph of a function has a tangent line and where the concavity changes.

### 2nd Derivative Test for Local Extremes

Suppose is continuous on an open interval that contains . Then, if and , the function has a local maximum at . If and , then has a local minimum at . If and , then may have a local maximum, a local minimum or neither at .

Exercise 4.4

1 – 8: Identify the inflection points and local maxima and minima (using both 1st and 2nd derivative tests). Sketch the graph of the function using behaviours.

1.

For critical point, .

The intervals are , and .

1st derivative test for local maxima or minima:

On , let .

, so the given function is increasing on ..

On , let .

, so the given function is decreasing on .

On , let .

, so the given function is increasing on .

At , changes sign from positive to negative. So, the given function has a local maxima at , with a value .

At , changes sign from negative to positive. So, the given function has a local minima at , with a value .

2nd derivative test for local maxima and minima:

At , , so the given function has a local maxima at , with a value .

At , , so the given function has a local minima at , with a value .

Test the concavity:

For concavity,

So, the intervals are and .

On , let .

so the given function is concave down on .

On , let .

so the given function is concave up on .

At , .

The point of inflection is .

## Indeterminant Forms and L’Hospital’s Rule

The forms , , , , , etc. are called indeterminant forms.

### L’Hospital’s Rule

Let and be continuous functions (on closed interval and containing ) and both functions be at . If and are differentiable on an open interval containing and on that interval if , then .

Example 1

form

L’Hospital’s Rule

Exercise 4.5

60.

form

form

Following L’Hospital’s Rule,

4.6

A rectangle has its base on the -axis and its upper two vertices on the parabola . What is the largest area the rectangle can have and what are its dimensions?

Let be a corner intersecting the parabola .

The base of the rectangle is and its height is .

The area of the rectangle,

For the critical point,

Of these, only lies in the interval .

At , .

At ,

At ,

The maximum area is at .

Base

Height

Example 5

Revenue,

Cost,

Is there a production level that maximizes profit? What is it?

At the critical point, .

At ,

At ,

Therefore, the maximum profit is at .

The maximum profit is .